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ON THE VANISHING VISCOSITY LIMIT IN A DISK

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Abstract. Let u be a solution to the Navier-Stokes equations in the unit disk with no-slip boundary conditions and viscosity $\nu > 0$, and let $\overline{u}$ be a smooth solution to the Euler equations. We say that the vanishing viscosity limit holds on $[0, T]$ if u converges to $\overline{u}$ in $L^\infty([0, T]; L^2)$. We show that a necessary and sufficient condition for the vanishing viscosity limit to hold is the vanishing with the viscosity of the time-space average of the energy of u in a boundary layer of width proportional to ν due to the modes (eigenfunctions of the Stokes operator) whose frequencies in the radial or the tangential direction lie between $L(\nu)$ and $M(\nu)$. Here, $L(\nu)$ must be of order less than $1/\nu$ and $M(\nu)$ must be of order greater than $1/\nu$.

1. Introduction

In the presence of a boundary, the question of whether solutions of the Navier-Stokes equations with no-slip boundary conditions converge to a solution of the Euler equations as the viscosity vanishes—the so-called vanishing viscosity limit—is very difficult. The convergence of most interest is of the velocities, uniformly over finite time and L^2 in space. Except in the very special case of radially symmetric initial vorticity in a disk, where convergence is known to hold (see Theorem 6.1), the question of convergence or the lack thereof is unresolved for nonzero initial velocity in a bounded domain. (For a half-space with analytic initial data, the vanishing viscosity limit is shown to hold in [14].)

Tosio Kato in [6] gave necessary and sufficient conditions on the velocity u of the Navier-Stokes equations for the vanishing viscosity limit to hold. The most interesting of these is that

$$\nu \int_0^T \|\nabla u(t)\|^2_{L^2(\Gamma_{c\nu})} \, dt \to 0 \text{ as } \nu \to 0,$$

where $\Gamma_{c\nu}$ is the boundary strip of width $c\nu$ with $c > 0$ fixed but arbitrary. Making only a small change to Kato's proof, it is possible to replace ∇u with the vorticity $\omega = \omega(u) = \partial_1 u^2 - \partial_2 u^1$, giving Equation (2.3) (see [7]). (The necessity of Equation (2.3) is immediate from Kato's condition, but because we do not have a boundary condition on the inner boundary of $\Gamma_{c\nu}$ the sufficiency of the condition requires proof.)

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Other necessary and sufficient conditions were established by Teman and Wang in [15] and [17]. These are the conditions in Equation (2.5) and Equation (2.6) of Theorem 2.3, and involve only the derivatives in the directions tangential to the boundary of either the tangential or normal components of the velocity, though for a slightly larger boundary layer. Finally, a condition that requires that the average energy density in the boundary layer of the same width as Kato's vanish with viscosity, Equation (2.7), is proven in [7]. All these conditions (which apply to a bounded domain in dimensions 2 and higher) are summarized in Theorem 2.3.

We consider the issue of vanishing viscosity in the (unit) disk and look for weaker necessary and sufficient conditions for the limit to hold. The reason for working in the disk is that the simple geometry allows us to make quite explicit calculations using the eigenfunctions of the Stokes operator, which are composed of Bessel functions of the first kind. In a sense, this connects the energy method with the geometry. What we find is that we need only consider certain ranges of frequencies (or equivalently, length scales) in the various conditions: this is Theorem 2.4. Although Theorem 2.4 is specific to the disk, there is no hydrodynamical reason to expect the disk to be special as regards the vanishing viscosity limit, so one would expect a version of the theorem to apply to all sufficiently smooth bounded domains in $\mathbb{R}^2$, and probably in higher dimensions as well. We discuss this issue more fully in Remark (2.1).

In [3], Cheng and Wang obtain a result regarding vanishing viscosity in two dimensions analogous to Equation (2.15) and Equation (2.16). Their result applies to an approximating sequence to a solution of the Navier-Stokes equations as the viscosity vanishes, whereas our result applies to the necessary and sufficient condition for the vanishing viscosity limit to hold. While for the other conditions in Theorem 2.4 we use very different techniques than those in [3], our proof of the necessity and sufficiency of Equation (2.15) and Equation (2.16) uses the key inequality in their paper. Section 7 contains a brief comparison between the two results.

In [13], the authors consider the Stokes problem (linearized Navier-Stokes equations) *external* to a disk with time-varying Dirichlet boundary conditions, showing that the vanishing viscosity limit holds. In fact, they do much more than this, giving an explicit construction of the solution to the Stokes problem and showing that it can be decomposed into the sum of the solution to the linearized Euler equations, the solution to the associated Prantdl equations, and a small correction term. The symmetry of the geometry allows the authors of [13] to construct the solutions in an explicit form (involving Bessel functions of the first and second kind). The nonlinear term in the Navier-Stokes equations makes an explicit solution impossible for us; however, we can expand the solution in terms of eigenfunctions of the Stokes operator for which we have an explicit form (in terms of Bessel functions of the first kind) which we can use to obtain finer estimates on the

behavior of the Navier-Stokes equations in the boundary layer than would be possible for a general domain.

A word on notation: We use C to represent an unspecified constant that always has the same value on both sides of an equality but may have a different value on each side of an inequality.

2. Definitions and Kato-type conditions

We now give definitions of the Euler and Navier-Stokes equations, and state the results from [6], [7], [15], and [17] that we will need.

In Section 4 we will specialize to the unit disk, but for now we assume only that Ω is a bounded domain in $\mathbb{R}^2$ with C^2-boundary Γ, and we let $\mathbf{n}$ be the outward normal vector to Γ.

A classical solution $(\overline{u}, \overline{p})$ to the Euler equations satisfies, for fixed $T > 0$,

$$(E) \quad \left\{ \begin{array}{l} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = \overline{f} \text{ and } \operatorname{div} \overline{u} = 0 \text{ on } [0,T] \times \Omega, \\ \overline{u} \cdot \mathbf{n} = 0 \text{ on } [0,T] \times \Gamma, \text{ and } \overline{u} = \overline{u}^0 \text{ on } \{0\} \times \Omega, \end{array} \right.$$

where $\operatorname{div} \overline{u}^0 = 0$. These equations describe the motion of an incompressible fluid of constant density and zero viscosity.

We assume that $\overline{u}^0$ is in $C^{k+\epsilon}(\Omega)$, $\epsilon > 0$, and that $\overline{f}$ is in $C^k([0,t] \times \Omega)$ for all $t > 0$, where $k = 1$ or 2. Then as shown in [8] (Theorem 1 and the remarks on p. 508-509), there exists a unique solution $\overline{u}$ in $C^1_{loc}([0,\infty); C^{k+\epsilon}(\Omega))$.

The Navier-Stokes equations describe the motion of an incompressible fluid of constant density and positive viscosity ν. A classical solution to the Navier-Stokes equations can be defined in analogy with (E) by

$$\left\{ \begin{array}{l} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f \text{ and } \operatorname{div} u = 0 \text{ on } [0,T] \times \Omega, \\ u = 0 \text{ on } [0,T] \times \Gamma, \text{ and } u = u^0_\nu \text{ on } \{0\} \times \Omega. \end{array} \right.$$

We will work, however, with weak solutions to the Navier-Stokes equations.

Definition 2.1 (Weak Navier-Stokes Solutions)**.** Given $T > 0$, viscosity $\nu > 0$, and initial velocity u^0_ν in H, u in $L^2([0,T]; V)$ with $\partial_t u$ in $L^2([0,T]; V')$ is a weak solution to the Navier-Stokes equations if $u(0) = u^0_\nu$ and

$$(NS) \qquad \int_\Omega \partial_t u \cdot v + \int_\Omega (u \cdot \nabla u) \cdot v + \nu \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v$$

for all v in V. (The spaces H and V are defined in Section 3.)

Definition 2.2. We say that the *vanishing viscosity limit holds* if

$$u \to \overline{u} \text{ in } L^\infty([0,T]; L^2(\Omega)) \text{ as } \nu \to 0. \tag{2.1}$$

Theorem 2.3 applies to a bounded domain with C^2-boundary in $\mathbb{R}^d$, $d \geq 2$. The conditions in Equation (2.2) and Equation (2.4) are due to Kato ([6]), the conditions in Equation (2.3) and Equation (2.7) appear in [7], and the conditions in Equation (2.5) and Equation (2.6) are due to Temam and Wang ([15], [17]).

Theorem 2.3. *Let $T > 0$ and assume that u_ν^0 is in H and that $\overline{u}^0$ is in $C^{k+\epsilon}(\Omega)$, $\epsilon > 0$ with $k = 1$ or 2. In addition, assume that*

(a) $u_\nu^0 \to \overline{u}^0$ *in* $L^2(\Omega)$ *as* $\nu \to 0$,
(b) f *is in* $L^1([0, T]; L^2(\Omega))$,
(c) $\|f - \overline{f}\|_{L^1([0,T];L^2(\Omega))} \to 0$ *as* $\nu \to 0$.

Let $\delta : [0, \infty) \to [0, \infty)$ be such that $\delta(\nu)$ converges to 0 while $\delta(\nu)/\nu$ diverges to ∞ as $\nu \to 0$. Then the the vanishing viscosity limit (Definition 2.2) holds if and only if any of the following conditions holds:

$$\nu \int_0^T \|\omega(s)\|_{L^2(\Omega)}^2 \, ds \to 0 \text{ as } \nu \to 0, \tag{2.2}$$

$$\nu \int_0^T \|\omega(s)\|_{L^2(\Gamma_{c\nu})}^2 \, ds \to 0 \text{ as } \nu \to 0, \tag{2.3}$$

$$\nu \int_0^T \|\nabla u(s)\|_{L^2(\Gamma_{c\nu})}^2 \, ds \to 0 \text{ as } \nu \to 0, \tag{2.4}$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 \, ds \to 0 \text{ as } \nu \to 0, \tag{2.5}$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\mathbf{n}}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 \, ds \to 0 \text{ as } \nu \to 0. \tag{2.6}$$

Here $\nabla_{\boldsymbol{\tau}}$ represents the derivatives in the boundary layer in the directions tangential to the boundary, $u_{\boldsymbol{\tau}}$ is the projection of u in the direction tangential to the boundary, and $u_{\mathbf{n}}$ is the projection of u in the direction normal to the boundary.

When $k = 2$, these conditions are also equivalent to

$$\frac{1}{\nu} \int_0^T \|u(s)\|_{L^2(\Gamma_{c\nu})}^2 \, ds \to 0 \text{ as } \nu \to 0. \tag{2.7}$$

The quantity in Equation (2.7) is proportional to the space-time average of the energy in the boundary layer.

We show (see Remark (5.2)) that in Equation (2.2), Equation (2.4), and Equation (2.7), contributions from the high frequency modes can be ignored. This result applies to an arbitrary bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a C^2-boundary.

Our main result is Theorem 2.4, which is an improvement of Theorem 2.3 in the special case of the unit disk. In what follows we decompose the solution u in the form

$$u(t, x) = \sum_{m=0}^\infty \sum_{j=1}^\infty g_{mj}(t) u_{mj}(x),$$

where (u_{mj}) are the eigenfunctions of the Stokes operator described in Section 3 and Section 4, and let

$$u^N(t,x) = \sum_{m=0}^{N} \sum_{j=1}^{N} g_{mj}(t) u_{mj}(x) \tag{2.8}$$

and

$$\widetilde{u}^N(t,x) = \sum_{m=0}^{N} \sum_{j=1}^{\infty} g_{mj}(t) u_{mj}(x) \tag{2.9}$$

with vorticities $\omega^N(t,x) = \omega(u^N(t,x))$ and $\widetilde{\omega}^N(t,x) = \omega(\widetilde{u}^N(t,x))$.

As we will see in Section 4, the frequency of u_{mk} in the tangential direction is m and the radial frequency of u_{mk} is, in effect, k. Thus, u^N includes the contributions from all modes with both frequencies less than N, while $\widetilde{u}^N$ includes the contributions from all modes with tangential frequency less than N.

Theorem 2.4. *Assume that Ω is the unit disk and make the same assumptions on the initial data, forcing, and the function δ as in Theorem 2.3. Let L and M be any functions mapping $(0,\infty)$ to $\mathbb{Z}^+$ with*

$$\nu L(\nu) \to 0,\ \nu M(\nu) \to \infty \text{ as } \nu \to 0. \tag{2.10}$$

Then the the vanishing viscosity limit (Definition 2.2) holds if and only if any of the following conditions holds:

$$\nu \int_0^T \|\omega(s)^{M(\nu)} - \omega^{L(\nu)}(s)\|^2_{L^2(\Omega)}\, ds \to 0 \text{ as } \nu \to 0, \tag{2.11}$$

$$\nu \int_0^T \|\omega(s) - \widetilde{\omega}^{L(\nu)}(s)\|^2_{L^2(\Omega)}\, ds \to 0 \text{ as } \nu \to 0, \tag{2.12}$$

$$\nu \int_0^T \|\omega(s) - \omega^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})}\, ds \to 0 \text{ as } \nu \to 0, \tag{2.13}$$

$$\nu \int_0^T \|\nabla u^{M(\nu)}(s) - \nabla u^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})}\, ds \to 0 \text{ as } \nu \to 0, \tag{2.14}$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}}(s) - \nabla_{\boldsymbol{\tau}} \widetilde{u}^{L(\delta)}_{\boldsymbol{\tau}}(s)\|^2_{L^2(\Gamma_{\delta(\nu)})}\, ds \to 0 \text{ as } \nu \to 0, \tag{2.15}$$

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} u_{\mathbf{n}}(s) - \nabla_{\boldsymbol{\tau}} \widetilde{u}^{L(\delta)}_{\mathbf{n}}(s)\|^2_{L^2(\Gamma_{\delta(\nu)})}\, ds \to 0 \text{ as } \nu \to 0. \tag{2.16}$$

When $k = 2$, these conditions are also equivalent to

$$\frac{1}{\nu} \int_0^T \|u^{M(\nu)}(s) - u^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})}\, ds \to 0 \text{ as } \nu \to 0. \tag{2.17}$$

Observe, for instance, that $u^{M(\nu)} - u^{L(\nu)}$ in Equation (2.17) represents the contribution from all modes whose frequencies in the radial or the tangential direction lie between $L(\nu)$ and $M(\nu)$.

Remark 2.1. By Lemma A.3 and Equation (4.7), u^N is essentially the contributions of all the modes with eigenvalues less than CN^2. In fact, suppose that we replace the definition of u^N in Equation (2.8) with

$$u^N(t,x) = \sum_{\{j : \lambda_j < N^2\}} g_j(t) u_j(x), \tag{2.18}$$

the single subscripts in Equation (2.18) referring to the eigenfunctions and eigenvalues of the Stokes operator on a general domain in $\mathbb{R}^d$, $d \geq 2$, defined in Section 3. It follows easily from Theorem 2.4 that the conditions in Equation (2.11), Equation (2.13), Equation (2.14), and Equation (2.17) continue to be equivalent to the vanishing viscosity limit. It is in this form that we would expect Theorem 2.4 to generalize to fairly arbitrary smooth domains in $\mathbb{R}^2$ and—with N^2 in Equation (2.18) replaced by N raised to some other power—to domains in $\mathbb{R}^d$, $d \geq 3$. The obstacle to establishing this generalization is the difficulty of obtaining the equivalents of Lemma A.8 and Lemma A.9—along with an approximate form of Lemma A.10—for high frequencies.

3. The Stokes operator in a bounded domain

Before specializing to the case of a disk, we discuss first some general properties related to the Stokes operator.

We define the function spaces H and V as follows (see Section I.1.4 of [16] for more details). First let

$$\mathcal{V} = \left\{ u \in (\mathcal{D}(\Omega))^2 : \operatorname{div} u = 0 \right\}$$

be the space of vector-valued divergence-free distributions on Ω. We let H be the closure of $\mathcal{V}$ in $L^2(\Omega)$ and V be the closure of $\mathcal{V}$ in $H^1_0(\Omega)$. Alternate characterizations of H and V are

$$\begin{aligned} H &= \left\{ u \in (L^2(\Omega))^2 : \operatorname{div} u = 0 \text{ in } \Omega,\ u \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}, \\ V &= \left\{ u \in (L^2(\Omega))^2 : \operatorname{div} u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \Gamma \right\}, \end{aligned}$$

the boundary conditions applying in terms of a trace.

By $\langle \cdot, \cdot \rangle$ we mean the inner product in $L^2(\Omega)$: $\langle f, g \rangle = \int_\Omega f \overline{g}$. (It will be convenient to use complex-valued eigenfunctions, so the complex conjugate is required in this definition. Our velocity fields and vorticities, however, are real, so conjugation will not always appear in our calculations.) Then $\langle u, v \rangle_H = \langle u, v \rangle$ and $\langle u, v \rangle_V = \langle \nabla u, \nabla v \rangle$.

Although $\mathcal{V}$ is dense in H it is not dense in $H \cap H^1(\Omega)$ (with the H^1-norm). Informally, this is because each element of $\mathcal{V}$ is zero on Γ and so the limit of a sequence of elements in $\mathcal{V}$ cannot become nonzero on the boundary

without the gradient near the boundary becoming indefinitely large. More formally, we have Lemma 3.1.

Lemma 3.1. *The space $\mathcal{V}$ is not dense in $H \cap H^1(\Omega)$.*

Proof. Let u be any element of V. Then its vorticity ω is in $L^2(\Omega) \subseteq L^1(\Omega)$ and must satisfy

$$\int_\Omega \omega = \int_\Omega \Delta\psi = \int_\Omega \operatorname{div} \nabla\psi = \int_\Gamma \nabla\psi \cdot \mathbf{n} = -\int_\Gamma u^\perp \cdot \mathbf{n} = 0, \tag{3.1}$$

where ψ is the stream function: $u = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$ and $\omega = \Delta\psi$. Because only $u \cdot \mathbf{n} = 0$ for u in H, the same cannot be said for u in $H \cap H^1(\Omega)$: Let u in $H \cap H^1(\Omega)$ have vorticity ω with nonzero total mass. Then for any sequence $\{v_j\}$ in $\mathcal{V}$,

$$\begin{aligned} \|\omega - \omega(v_j)\|_{L^2(\Omega)} &\geq C\,\|\omega - \omega(v_j)\|_{L^1(\Omega)} \geq C\left|\int_\Omega (\omega - \omega(v_j))\right| \\ &= C\left|\int_\Omega \omega\right| > 0, \end{aligned}$$

so $\mathcal{V}$ cannot be dense in $H \cap H^1(\Omega)$. □

We now briefly describe the properties we will need of the Stokes operator A on Ω, referring the reader, for instance, to Section I.2 of [16] for more details. One way to define A is that given u in $V \cap H^2(\Omega)$, Au in H satisfies $Au = -\Delta u + \nabla p$ for some harmonic scalar field p. We have $D(A) = V \cap H^2(\Omega)$ with A mapping $D(A)$ onto H, and there exists a set of eigenfunctions $\{u_j\}$ for A, complete in H and in V, with corresponding eigenvalues $\{\lambda_j\}$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and each u_j is in $H^2(\Omega)$ since we are assuming that Γ is C^2. (When we specialize to the disk, the eigenfunctions will be in $C^\infty(\Omega)$.) An eigenfunction u_j of A satisfies $Au_j = \lambda_j u_j$ or, equivalently,

$$\begin{cases} \Delta u_j + \lambda_j u_j = \nabla p_j,\ \Delta p_j = 0,\ \operatorname{div} u_j = 0, \text{ on } \Omega, \\ u_j = 0 \text{ on } \Gamma. \end{cases} \tag{3.2}$$

The eigenfunctions are orthogonal in both H and V. The usual convention is to make the eigenvectors orthonormal in H, but we will find it more convenient to normalize them to be orthonormal in V so that $\|\nabla u_j\|^2_{L^2(\Omega)} = \|\omega_j\|^2_{L^2(\Omega)} = 1$ and

$$\|u_j\|^2_{L^2(\Omega)} = \langle u_j, u_j \rangle = \frac{1}{\lambda_j}\langle u_j, Au_j \rangle = \frac{1}{\lambda_j}\langle \nabla u_j, \nabla u_j \rangle = \frac{1}{\lambda_j}. \tag{3.3}$$

Moreover, we have Lemma 3.2.

Lemma 3.2. *If u is in V with $\omega = \omega(u)$ then*

$$u = \sum_{j=1}^\infty \langle \omega, \omega_j \rangle u_j, \tag{3.4}$$

with the sum converging in both V and H.

Proof. Let u be in V and let $u^n = \sum_{j=1}^n (\langle u, u_j \rangle_H / \langle u_j, u_j \rangle_H) u_j$. Then u^n converges in H to u because $\{u_j\}$ is complete in H. But,

$$\begin{aligned}\frac{\langle u, u_j \rangle_H}{\langle u_j, u_j \rangle_H} &= \frac{\lambda_j \langle u, u_j \rangle}{\lambda_j \langle u_j, u_j \rangle} = \frac{\langle u, A u_j \rangle}{\langle u_j, A u_j \rangle} = \frac{\langle \nabla u, \nabla u_j \rangle}{\langle \nabla u_j, \nabla u_j \rangle} \\ &= \langle \nabla u, \nabla u_j \rangle = \langle \omega, \omega_j \rangle,\end{aligned}$$

so the expansion of u in V in terms of the eigenfunctions of A is the same as the expansion of u in H (and the coefficients are as given in Equation (3.4)), meaning that u^n converges in V to u as well. □

In the proof of Lemma 3.2 we used the identity $\langle \nabla u, \nabla v \rangle = \langle \omega(u), \omega(v) \rangle$ for all u, v in V, which follows by integrating by parts. Were we to use the definition of $\omega(u)$ as the antisymmetric matrix $(\nabla u - (\nabla u)^T)/2$, which is usual in higher dimensions, this would have introduced a factor of 2 into Equation (3.4).

Corollary 3.3. *If u is in V then*

$$\nabla u = \sum_{j=1}^\infty \langle \omega, \omega_j \rangle \nabla u_j \text{ and } \omega = \sum_{j=1}^\infty \langle \omega, \omega_j \rangle \omega_j,$$

with the sums converging in $L^2(\Omega)$.

Since the solution u to (NS) lies in V for all positive time, we can write

$$\begin{aligned}\omega(t) &= \sum_{j=1}^\infty g_j(t) \omega_j, \quad u(t) = \sum_{j=1}^\infty g_j(t) u_j, \\ \|\omega(t)\|^2_{L^2(\Omega)} &= \sum_{j=1}^\infty |g_j(t)|^2, \quad \|u(t)\|^2_{L^2(\Omega)} = \sum_{j=1}^\infty \frac{|g_j(t)|^2}{\lambda_j},\end{aligned} \tag{3.5}$$

where g_j are functions of time. The expansion of u will converge for all $t \geq 0$ and that of ω for $t > 0$—and also for $t = 0$ if and only if the initial velocity is in V; in general, we only assume that it is H. Because $u(t) \to u^0_\nu$ in $L^2(\Omega)$ as $t \to 0$, each $g_j(t)$ is continuous at $t = 0$, though this does not mean that $\omega(t)$ is continuous in $L^2(\Omega)$ at $t = 0$. Also, note that $g_j(t)$ is complex-valued since the eigenvectors are complex-valued, but $u(t)$ and $\omega(t)$ are real-valued.

4. Eigenfunctions of the Stokes operator in the unit disk

We now fix Ω to be the unit disk in $\mathbb{R}^2$ centered at the origin.

In [10], a complete set of eigenfunctions for the annulus is derived in terms of Bessel functions of the first and second kind, J_n and Y_n. By ignoring the terms involving Y_n and modifying somewhat the calculation of the eigenvalues, one can easily obtain the eigenfunctions for a disk. We will, however, derive the vorticity of the eigenfunctions directly, as this is quite easy. In determining the eigenvalues and the velocity of the eigenfunctions, which is more difficult, we will rely on the results in [10].

Taking the curl of Equation (3.2) (with $u = u_j$), we see that the vorticity $\omega = \omega(u)$ satisfies

$$\begin{cases} \Delta\omega + \lambda\omega = 0 \text{ on } \Omega, \\ u = 0 \text{ on } \Gamma. \end{cases} \tag{4.1}$$

That is, ω is an eigenfunction of the negative Laplacian, but with boundary conditions on the velocity u.

Ignoring for the moment the issue of boundary conditions, we use separation of variables to look for a complete set of solutions to $\Delta\omega + \lambda\omega = 0$ on Ω. Writing

$$\omega(r, \theta) = f_n(r)e^{in\theta}$$

in polar coordinates, $n = 0, 1, 2, \dots$, and using

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

gives

$$\left(\frac{d^2 f_n}{dr^2} + \frac{1}{r}\frac{df_n}{dr} + \left(\lambda - \frac{n^2}{r^2}\right) f_n\right) e^{in\theta} = 0. \tag{4.2}$$

Since J_n, the Bessel function of the first kind of order n, is a solution of

$$\frac{d^2 J_n(s)}{ds^2} + \frac{1}{s}\frac{dJ_n(s)}{ds} - \left(1 - \frac{n^2}{s^2}\right) J_n(s) = 0, \tag{4.3}$$

making the change of variables $s = \lambda^{1/2} r$, we see that Equation (4.2) holds with $f_n(r) = J_n(\lambda^{1/2} r)$. Thus, the eigenfunctions have vorticity of the form $J_n(\lambda^{1/2} r)e^{in\theta}$ and it remains to determine the eigenvalues that satisfy $u = 0$ on the boundary.

The easiest way to do this is to use the expressions in [10]. For $n = 0$, we drop the term involving Y_1 in Equation (30) p. 406 of [10], giving

$$u_{0k}(r, \theta) = \lambda_{0k}^{-1/2} J_1(\lambda_{0k}^{1/2} r)\widehat{e}_\theta,$$

where $\lambda_{0k}^{1/2}$, $k = 1, 2, \dots$, are the eigenvalues described below. For $n \geq 1$, dropping the terms involving the Bessel functions of the second kind from the last equation on p. (406) of [10], we have

$$\begin{aligned} u_{nk}(r, \theta) = {} & \left(\frac{in}{\lambda_{nk} r} J_n(\lambda_{nk}^{1/2} r) + \frac{D_{nk} in^2}{\lambda_{nk}^2} r^{n-1}\right) e^{in\theta}\widehat{e}_r \\ & + \left(\frac{1}{2\lambda_{nk}^{1/2}}\left(J_{n+1}(\lambda_{nk}^{1/2} r) - J_{n-1}(\lambda_{nk}^{1/2} r)\right) - \frac{D_{nk} n^2}{\lambda_{nk}^2} r^{n-1}\right) e^{in\theta}\widehat{e}_\theta, \end{aligned}$$

the eigenvalues λ_{nk}, $k = 1, 2, \dots$, being described below and the D_{nk} being undetermined constants. In both cases we scaled the eigenfunctions differently than in [10]. A direct calculation shows that

$$\omega_{nk}(r, \theta) \stackrel{\text{def}}{=} \omega(u_{nk})(r, \theta) = C_{nk} J_n(\lambda_{nk}^{1/2} r)e^{in\theta},$$

where C_{nk} is a normalization constant. A direct calculation also shows that $\operatorname{div} u_{nk} = 0$.

For $n = 0$, $\lambda_{0k}^{1/2} = j_{1k}$, where

$$j_{nk} \text{ is the } k\text{-}th \text{ positive root of } J_{n+1}(x) = 0, \tag{4.4}$$

as this gives $u_{0k}(1, \theta) = 0$. Setting $u_{nk}(1, \theta) = 0$ we obtain two equations in the two unknowns λ_{nk} and D_{nk}. We eliminate D_{nk} from the two equations to obtain a single equation for λ_{nk}. Then using the identity in Equation (A.2) we obtain the equation

$$\lambda_{nk}^{1/2} J_n'(\lambda_{nk}^{1/2}) - n J_n(\lambda_{nk}^{1/2}) = 0.$$

Thus, $\lambda_{nk}^{1/2}$ is the k-th positive root of

$$x J_n'(x) - n J_n(x) = -x J_{n+1}(x) = 0, \tag{4.5}$$

$k = 1, 2, \dots$, where we used Equation (A.4). That is, $\lambda_{nk}^{1/2} = j_{n+1,k}$. It follows then that

$$D_{nk} = -\frac{\lambda_{nk} J_n(\lambda_{nk}^{1/2})}{n} = -\frac{j_{n+1,k}^2 J_n(j_{n+1,k})}{n}. \tag{4.6}$$

Since we are normalizing the eigenfunctions so that $\langle \omega_{mj}, \omega_{nk} \rangle = \delta_{mn} \delta_{jk}$, we must choose C_{nk} so that

$$\begin{aligned} C_{nk}^{-2} &= \|J_n(j_{n+1,k} r) e^{in\theta}\|_{L^2(\Omega)}^2 = 2\pi \int_0^1 r J_n(j_{n+1,k} r)^2 \, dr \\ &= 2\pi \frac{r^2}{2} \left[J_n(j_{n+1,k} r)^2 - J_{n-1}(j_{n+1,k} r) J_{n+1}(j_{n+1,k} r) \right]_0^1 \\ &= \pi J_n(j_{n+1,k})^2. \end{aligned}$$

Here we used Equation (A.9).

To summarize, the vorticity of the eigenfunctions is given by

$$\omega_{nk}(r, \theta) = C_{nk} J_n(j_{n+1,k} r) e^{in\theta},$$

with eigenvalue

$$\lambda_{nk} = j_{n+1,k}^2, \tag{4.7}$$

and where

$$C_{nk} = \frac{1}{\pi^{1/2} |J_n(j_{n+1,k})|},$$

$n = 0, 1, \dots$, $k = 1, 2, \dots$. With our choice of normalization of the eigenfunctions (Equation (3.3)), the velocity becomes

$$\begin{aligned} u_{nk}(r, \theta) = {} & \frac{J_n(\alpha r) - J_n(\alpha) r^n}{\pi^{1/2} \alpha^2 \, |J_n(\alpha)| \, r} i n e^{in\theta} \widehat{e}_r \\ & + \frac{\alpha \left(J_{n+1}(\alpha r) - J_{n-1}(\alpha r) \right) + 2n J_n(\alpha) r^{n-1}}{2\pi^{1/2} \alpha^2 \, |J_n(\alpha)|} e^{in\theta} \widehat{e}_\theta, \end{aligned} \tag{4.8}$$

where $\alpha = j_{n+1,k}$.

5. Proof of Theorem 2.4

From the fundamental energy equality for (NS) we have for all t in $[0,T]$,

$$\nu \int_0^t \|\nabla u\|^2_{L^2(\Omega)} = \nu \int_0^t \|\omega\|^2_{L^2(\Omega)} \le \frac{1}{2}\|u^0_\nu\|^2_H + 4\,\|f\|^2_{L^1([0,T];L^2(\Omega))}\,.$$

It follows from Equation (3.5) and assumptions (a) and (b) of Theorem 2.3 that for all sufficiently small $\nu > 0$,

$$\nu \int_0^t \|\omega\|^2_{L^2(\Omega)} = \nu \int_0^t \sum_{m=0}^\infty \sum_{j=1}^\infty |g_{mj}(s)|^2 \, ds \le C. \tag{5.1}$$

Theorem 5.1. *With the assumptions of Theorem 2.4,*

$$\lim_{\nu \to 0} \nu \int_0^t \|\omega^{L(\nu)}\|^2_{L^2(\Gamma_{c\nu})} = 0. \tag{5.2}$$

Proof. Using Lemma A.10,

$$\begin{aligned}
&\nu \int_0^t \|\omega^{L(\nu)}\|^2_{L^2(\Gamma_{c\nu})} \\
&\quad = \nu \int_0^t \sum_{m=0}^{L(\nu)} \sum_{j=1}^{L(\nu)} \sum_{n=0}^{L(\nu)} \sum_{k=1}^{L(\nu)} g_{mj}(s)\overline{g_{nk}(s)}\, ds\, \langle \omega_{mj}, \omega_{nk} \rangle_{L^2(\Gamma_{c\nu})} \\
&\quad = \nu \int_0^t \sum_{n=0}^{L(\nu)} \sum_{j=1}^{L(\nu)} \sum_{k=1}^{L(\nu)} g_{nj}(s)\overline{g_{nk}(s)}\, ds\, \langle \omega_{nj}, \omega_{nk} \rangle_{L^2(\Gamma_{c\nu})} \\
&\quad \le \nu \int_0^t \sum_{n=0}^{L(\nu)} \sum_{j=1}^{L(\nu)} \sum_{k=1}^{L(\nu)} |g_{nj}(s)|\, |g_{nk}(s)| \; ds\, \|\omega_{nj}\|_{L^2(\Gamma_{c\nu})} \|\omega_{nk}\|_{L^2(\Gamma_{c\nu})} \\
&\quad = \nu \int_0^t \sum_{n=0}^{L(\nu)} \left(\sum_{j=1}^{L(\nu)} |g_{nj}(s)|\, \|\omega_{nj}\|_{L^2(\Gamma_{c\nu})} \right)^2 ds \\
&\quad \le \nu \int_0^t \sum_{n=0}^{L(\nu)} \sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2 \sum_{j=1}^{L(\nu)} \|\omega_{nj}\|^2_{L^2(\Gamma_{c\nu})} \, ds,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last step.

By Lemma A.3,

$$1/L(\nu) < C/(L(\nu)+2) \le C/j_{L(\nu)+1,1} = C\lambda^{-1/2}_{L(\nu),1}.$$

Since $\nu L(\nu) \to 0$ as $\nu \to 0$, for all sufficiently small ν we have $c\nu < C/L(\nu) \le 2\pi\lambda^{-1/2}_{L(\nu),1} \le 2\pi\lambda^{-1/2}_{j,1}$ for all $j \le L(\nu)$. So by Lemma A.8,

$$\sum_{j=1}^{L(\nu)} \|\omega_{nj}\|^2_{L^2(\Gamma_{c\nu})} \le C\nu L(\nu). \tag{5.3}$$

Then using Equation (5.1),

$$\nu \int_0^t \|\omega^{L(\nu)}\|^2_{L^2(\Gamma_{c\nu})} \le C\nu L(\nu) \left(\nu \int_0^t \sum_{n=0}^{L(\nu)} \sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2 \, ds \right) \le C\nu L(\nu),$$

which vanishes with ν by the condition in Equation (2.10), and Equation (5.2) therefore holds. □

Remark 5.1. We could try to improve Theorem 5.1 by using $\omega(\widetilde{u}^N)$ of Equation (2.9) in place of ω^N, thereby incorporating all of the frequencies in the radial direction for a given angular frequency. Unfortunately, the best bound that one can achieve on $\|\omega_{nj}\|^2_{L^2(\Gamma_\delta)}$ for $j > n$ is the extension of Lemma A.8 described in Remark (A.1), and this is very much insufficient to bound the terms with $j > n$.

Another possible approach is to try to incorporate the destructive interference that occurs in the inner product of two eigenfunctions in the boundary layer that the use of Hölder's inequality in our proof of Theorem 5.1 ignored. The best bound one can hope to obtain is that

$$|\left\langle \omega_{nj}, \omega_{nk} \right\rangle_{L^2(\Gamma_\delta)}| \le \frac{C}{|k - j|}$$

for all δ in $[0, 1]$ and without restriction on n, j, or k except that $j \ne k$. We could then follow the obvious approach of decomposing the equivalent of the first sum in the proof of Theorem 5.1 into four pieces: a diagonal term where $m = n$ and $j = k$ and three terms containing low frequencies in j and k, low frequencies in j and high frequencies in k, and high frequencies in both j and k. If we do this, however, we will find that the factor of $1/|k - j|$ is just insufficient to obtain convergence.

Corollary 5.2. *The conditions in Equation (2.1) and Equation (2.13) of Theorem 2.4 are equivalent.*

Proof. That Equation (2.1) implies Equation (2.13) follows directly from Theorem 2.3. So assume that Equation (2.13) holds. Because $\|A + B\|^2 \le 2\|A\|^2 + 2\|B\|^2$ for any norm,

$$\nu \int_0^t \|\omega\|^2_{L^2(\Gamma_{c\nu})} \le 2\nu \int_0^t \|\omega^{L(\nu)}\|^2_{L^2(\Gamma_{c\nu})} + 2\nu \int_0^t \|\omega - \omega^{L(\nu)}\|^2_{L^2(\Gamma_{c\nu})}.$$

This vanishes with ν by Theorem 5.1 and Equation (2.13), showing that Equation (2.3) holds and hence by Theorem 2.3 that Equation (2.1) holds. □

Theorem 5.3. *With the assumptions of Theorem 2.4,*

$$\lim_{\nu \to 0} \frac{1}{\nu} \int_0^t \|u(s) - u^{M(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})} \, ds = 0 \tag{5.4}$$

and

$$\lim_{\nu \to 0} \frac{1}{\nu} \int_0^t \|u^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})} \, ds = 0. \tag{5.5}$$

Proof. We can write $u(t) - u^{M(\nu)}(t) = A(t) + B(t)$, where

$$A(t) = \sum_{m=1}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} g_{mj}(t) u_{mj}(x), \quad B(t) = \sum_{m=M(\nu)+1}^{\infty} \sum_{j=1}^{\infty} g_{mj}(t) u_{mj}(x)$$

and

$$\|u(t) - u^{M(\nu)}(t)\|^2_{L^2(\Gamma_{c\nu})} \le 2 \|A(t)\|^2_{L^2(\Gamma_{c\nu})} + 2 \|B(t)\|^2_{L^2(\Gamma_{c\nu})}.$$

Now,

$$\begin{aligned}
\|A(t)\|^2_{L^2(\Gamma_{c\nu})} \le \|A(t)\|^2_{L^2(\Omega)} &= \sum_{m=1}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} |g_{mj}(t)|^2 \|u_{mj}\|^2_{L^2(\Omega)} \\
&= \sum_{m=1}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} \frac{|g_{mj}(t)|^2}{\lambda_{mj}} \le \frac{1}{\lambda_{1M(\nu)}} \sum_{m=1}^{M(\nu)} \sum_{j=M(\nu)+1}^{\infty} |g_{mj}(t)|^2 \\
&\le \frac{1}{\lambda_{1M(\nu)}} \|\omega(t) - \omega^{M(\nu)}(t)\|^2_{L^2(\Omega)},
\end{aligned}$$

where we used Equation (3.3). Similarly,

$$\|B(t)\|^2_{L^2(\Gamma_{c\nu})} \le \|B(t)\|^2_{L^2(\Omega)} \le \frac{1}{\lambda_{M(\nu)1}} \|\omega(t) - \omega^{M(\nu)}(t)\|^2_{L^2(\Omega)}.$$

By Equation (4.7) and Lemma A.3, $\lambda_{M(\nu)1}$ and $\lambda_{1M(\nu)}$ are both bounded below (and above) by $CM(\nu)^2$, so

$$\|u(t) - u^{M(\nu)}(t)\|^2_{L^2(\Gamma_{c\nu})} \le \frac{C}{M(\nu)^2} \|\omega(t) - \omega^{M(\nu)}(t)\|^2_{L^2(\Omega)}.$$

Then,

$$\begin{aligned}
&\frac{1}{\nu} \int_0^t \|u(s) - u^{M(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})} \, ds \\
&\quad \le \frac{C}{\nu M(\nu)^2} \int_0^t \|\omega(s) - \omega^{M(\nu)}(s)\|^2_{L^2(\Omega)} \, ds \\
&\quad \le \frac{C}{\nu M(\nu)^2} \int_0^t \|\omega(s)\|^2_{L^2(\Omega)} \, ds \\
&\quad = \frac{C}{\nu^2 M(\nu)^2} \nu \int_0^t \|\omega(s)\|^2_{L^2(\Omega)} \, ds \le \frac{C}{\nu^2 M(\nu)^2},
\end{aligned}$$

where in the last inequality we used Equation (5.1). This vanishes with ν by the assumption on M in Equation (2.10) giving Equation (5.4).

Arguing as in the proof of Theorem 5.1,

$$\begin{aligned}\frac{1}{\nu}\int_0^t \|u^L(s)\|^2_{L^2(\Gamma_{c\nu})}\,ds &\le \frac{1}{\nu}\int_0^t \sum_{n=0}^{L(\nu)}\sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2 \sum_{j=1}^{L(\nu)} \|u_{nj}\|^2_{L^2(\Gamma_{c\nu})}\,ds \\ &\le \frac{CL(\nu)\nu^3}{\nu}\int_0^t \sum_{n=0}^{L(\nu)}\sum_{j=1}^{L(\nu)} |g_{nj}(s)|^2\,ds \le CL(\nu)\nu\end{aligned}$$

for all sufficiently small ν. In the second inequality we used Lemma A.9 and in the last inequality we used Equation (5.1). This integral also vanishes with ν by the assumption on L in Equation (2.10) giving Equation (5.5). $\square$

Corollary 5.4. *The conditions in Equation (2.1), Equation (2.11), Equation (2.14), and Equation (2.17) of Theorem 2.4 are equivalent.*

Proof. For sufficiently large ν, $L(\nu) \le M(\nu)$, and we have

$$\begin{aligned}\|u(s)\|^2_{L^2(\Gamma_{c\nu})} &\le 3\|u^{M(\nu)}(s) - u^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})} \\ &\quad + 3\|u^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})} + 3\|u(s) - u^{M(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})}.\end{aligned}$$

It follows from Theorem 5.3 that

$$\limsup_{\nu\to 0}\frac{1}{\nu}\int_0^t \|u(s)\|^2_{L^2(\Gamma_{c\nu})} \le 3\limsup_{\nu\to 0}\frac{1}{\nu}\int_0^t \|u^{M(\nu)} - u^{L(\nu)}\|^2_{L^2(\Gamma_{c\nu})}.$$

In particular, the first limsup is zero if and only if the second limsup is zero (the reverse inequality without the factor of 3 being trivial). Then Equation (2.7) of Theorem 2.3 shows that Equation (2.17) holds if and only if Equation (2.1) holds. The sufficiency of Equation (2.11) and Equation (2.14) for Equation (2.1) to hold then follows from Poincaré's inequality in the form

$$\begin{aligned}\|u^{M(\nu)}(s) - u^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})} &\le C\nu^2\|\nabla u^{M(\nu)}(s) - \nabla u^{L(\nu)}(s)\|^2_{L^2(\Gamma_{c\nu})} \\ &\le C\nu^2\|\nabla u^{M(\nu)}(s) - \nabla u^{L(\nu)}(s)\|^2_{L^2(\Omega)} \\ &= C\nu^2\|\omega^{M(\nu)}(s) - \omega^{L(\nu)}(s)\|^2_{L^2(\Omega)}.\end{aligned}$$

The necessity of Equation (2.11) and Equation (2.14) follow immediately from Theorem 2.3. $\square$

Remark 5.2. If we replace the definition of u^N in Equation (2.8) with that in Equation (2.18), then it is clear that Equation (5.4) continues to hold in any bounded domain in $\mathbb{R}^2$ with a C^2-boundary. It follows as in Corollary 5.4 that the vanishing viscosity limit of Definition 2.2 holds if and only if the condition in Equation (2.11), Equation (2.14), or (when $k = 2$) Equation (2.17) holds with the term involving $u^{L(\nu)}$ in each of these conditions removed. A similar result would hold in any dimension for an arbitrary bounded domain with a C^2-boundary.

Theorem 5.5. *With the assumptions of Theorem 2.4,*

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{\tau}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 \, ds \to 0 \text{ as } \nu \to 0 \tag{5.6}$$

and

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} \widetilde{u}_{\mathbf{n}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 \, ds \to 0 \text{ as } \nu \to 0. \tag{5.7}$$

Proof. In the unit disk, $u_{\boldsymbol{\tau}} = u^\theta$ and $\nabla_{\boldsymbol{\tau}} = \partial_\sigma$, where σ is arc length along the circle of radius r, in which r is held constant. Thus,

$$\nabla_{\boldsymbol{\tau}} u_{\boldsymbol{\tau}} = \frac{\partial u^\theta}{\partial \sigma} = \frac{1}{r}\frac{\partial u^\theta}{\partial \theta}$$

and for any positive integer N it follows from Poincaré's inequality that

$$\begin{aligned} \|\nabla_{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{\tau}}^N(s)\|_{L^2(\Gamma_\delta)}^2 &= \left\| \frac{1}{r}\frac{\partial}{\partial\theta}(\widetilde{u}^N(s))^\theta \right\|_{L^2(\Gamma_\delta)}^2 \le \frac{1}{(1-\delta)^2} \left\| \frac{\partial}{\partial\theta}(\widetilde{u}^N(s))^\theta \right\|_{L^2(\Gamma_\delta)}^2 \\ &\le \frac{C\delta^2}{(1-\delta)^2} \left\| \frac{\partial^2}{\partial r \partial\theta}(\widetilde{u}^N(s))^\theta \right\|_{L^2(\Gamma_\delta)}^2 \le \frac{C\delta^2}{(1-\delta)^2} \left\| \frac{\partial^2}{\partial r \partial\theta}(\widetilde{u}^N(s))^\theta \right\|_{L^2(\Omega)}^2 . \end{aligned}$$

But,

$$\begin{aligned} \frac{\partial^2}{\partial r \partial\theta}(\widetilde{u}^N(s))^\theta &= \sum_{m=0}^N \sum_{j=1}^\infty g_{mj}(s) \frac{\partial^2}{\partial r \partial \theta} u_{mj}(r,\theta) \\ &= i \sum_{m=0}^N m \sum_{j=1}^\infty g_{mj}(s) \frac{\partial}{\partial r} u_{mj}(r,\theta), \end{aligned}$$

the last equality following from the simple dependence of u_{mj} on θ in Equation (4.8). Thus,

$$\begin{aligned} \left\| \frac{\partial^2}{\partial r \partial\theta}(\widetilde{u}^N(s))^\theta \right\|_{L^2(\Omega)}^2 &\le \left\| i \sum_{m=0}^N m \sum_{j=1}^\infty g_{mj}(s) \nabla u_{mj}(r,\theta) \right\|_{L^2(\Omega)}^2 \\ &= \sum_{m=0}^N m^2 \sum_{j=1}^\infty |g_{mj}(s)|^2 \le N^2 \sum_{m=0}^N \sum_{j=1}^\infty |g_{mj}(s)|^2 \le N^2 \|\nabla u\|_{L^2(\Omega)}^2, \end{aligned}$$

where we used the orthonormality of the eigenfunctions in V.

Combining these two inequalities gives

$$\|\nabla_{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{\tau}}^N(s)\|_{L^2(\Gamma_\delta)}^2 \le \frac{CN^2\delta^2}{(1-\delta)^2} \|\nabla u\|_{L^2(\Omega)}^2.$$

Then using Equation (5.1),

$$\nu \int_0^T \|\nabla_{\boldsymbol{\tau}} \widetilde{u}_{\boldsymbol{\tau}}^{L(\delta)}(s)\|_{L^2(\Gamma_{\delta(\nu)})}^2 \le \frac{CL(\delta)^2\delta^2}{(1-\delta)^2} \nu \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 \le \frac{CL(\delta)^2\delta^2}{(1-\delta)^2}.$$

This vanishes with δ by the assumption on L in Equation (2.10) and hence vanishes with ν since δ vanishes with ν, giving Equation (5.6). The proof of Equation (5.7) is entirely analogous. □

The technique used in the proof of Theorem 5.5 comes from the key inequality following Equation (3.21) in [3].

Corollary 5.6. *The conditions in Equation (2.1), Equation (2.12), Equation (2.15), and Equation (2.16) of Theorem 2.4 are equivalent.*

Proof. This corollary can be proved much along the lines of the proofs of Corollary 5.2 and Corollary 5.4. (It is here that we use the assumption that $\delta(\nu)/\nu$ diverges to ∞ as $\nu \to 0$, which is needed in applying Theorem 2.3.) □

Together, Corollary 5.2, Corollary 5.4, and Corollary 5.6 establish Theorem 2.4.

6. Radially symmetric initial vorticity

It follows from Lemma 3.1 that if an initial velocity, no matter how smooth, lies in H but not in V and has a vorticity whose total mass is nonzero, then the velocity of the corresponding solution to (NS) will be discontinuous in $H^1(\Omega)$ at time zero. In the same way, the vanishing viscosity limit of the vorticity cannot hold in $L^\infty([0,T]; L^2(\Omega))$ for such an initial velocity, since the total mass of the vorticity for the solution to (E) is conserved over time. This means that we might expect a different character to the vanishing viscosity limit when the initial vorticity has zero total mass versus when it has nonzero total mass.

Indeed, this is what happens in the special case of radially symmetric initial vorticity where, when the initial velocity is in V (which is equivalent for radially symmetric vorticity to the total mass of the vorticity being zero) we obtain convergence in $L^\infty([0,T]; L^2(\Omega))$ of both the velocity and the vorticity (see [1]), whereas for initial velocity in H we obtain only the convergence of the velocity in this space, as in Theorem 6.1. Such convergence follows immediately from the conditions in Equation (2.12), Equation (2.15), or Equation (2.16) of Theorem 2.4. The convergence also follows from the sufficiency of the conditions in Equation (2.5) and Equation (2.6) as established in [15], since both conditions are satisfied (the gradients in the tangential direction being zero) as pointed out in [17]. When the forcing is zero, however, there is a simple proof that uses only Kato's original conditions.

Theorem 6.1. *Assume that u_ν^0 and $\overline{u}^0$ are as in Theorem 2.3 with (for simplicity) $u_\nu^0 = \overline{u}^0$, that $f = \overline{f} = 0$, and that $\omega^0 = \omega(u^0)$ is radially symmetric. Then the vanishing viscosity limit of Equation (2.1) holds.*

Proof. Because ω^0 is radially symmetric, ω remains radially symmetric for all time, so $\omega(u \cdot \nabla u) = u \cdot \nabla \omega = 0$. Then because Ω is simply connected, $u \cdot \nabla u =$

∇q for some scalar field q, and the nonlinear term in (NS) disappears. Thus, (NS) reduces to $u_\nu(0) = \overline{u}^0$ and

$$\int_\Omega \partial_t u_\nu \cdot v + \nu \int_\Omega \nabla u_\nu \cdot \nabla v = 0 \tag{6.1}$$

for all v in V. This is the heat equation in weak form, which is invariant under the transformation $(\nu, t, x) \mapsto (1, \nu t, x)$. That is, if u_1 is a solution to Equation (6.1) with $\nu = 1$, then $u_\nu(t, x) = u_1(\nu t, x)$ is a solution to Equation (6.1) because $u_\nu(0) = u_1(0) = \overline{u}^0$ and

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t} u_1(\nu t, x) \cdot v(x)\, dx + \nu \int_\Omega \nabla u_1(\nu t, x) \cdot \nabla v(x)\, dx \\
&\qquad = \nu \left[\int_\Omega (\partial_t u_1)(\nu t, x) \cdot v(x)\, dx + \int_\Omega \nabla u_1(\nu t, x) \cdot \nabla v(x)\, dx \right] = 0.
\end{aligned}$$

It follows that

$$\nu \int_0^t \|\omega(s)\|_{L^2(\Omega)}^2 \, ds = \nu \int_0^t \|\omega_1(\nu s)\|_{L^2(\Omega)}^2 \, ds = \int_0^{\nu t} \|\omega_1(\tau)\|_{L^2(\Omega)}^2 \, d\tau.$$

This vanishes as $\nu \to 0$ by the continuity of the integral, because u_1 is in $L^2([0, T]; V)$. The limit in Equation (2.1) then follows from the condition in Equation (2.2) of Theorem 2.3. □

The proof of of Theorem 6.1 does not yield a bound on the rate of convergence in Equation (2.1). Also, without assuming that the initial vorticity is radially symmetric, the argument in the proof of Theorem 6.1 can be applied to solutions to the Stokes problem (the linearized Navier-Stokes equations) to show that they converge in the vanishing viscosity limit to a solution to the linearized Euler equations (which is just the steady state solution $\overline{u} = \overline{u}^0$). This would be more interesting, though, if time-varying Dirichlet boundary conditions, for instance, could be incorporated, as in [13].

In the simpler case of u_ν^0 also lying in V, the solution to (E), which is steady state, is zero on the boundary. This eliminates the troublesome boundary term that appears in the direct energy argument bounding $\|u(t) - \overline{u}(t)\|_{L^2(\Omega)}$, giving an extremely simple proof of Equation (2.1). We give, however, a longer proof of convergence using Theorem 2.3, because it suggests how we might treat more general initial velocities. For convenience, we assume zero forcing and more regularity on the initial velocity than is strictly necessary.

Theorem 6.2. *Assume that ω_ν^0 is radially symmetric, $u_\nu^0 = \overline{u}^0$ is in $H^3(\Omega) \cap V$, and there is no forcing. Then the vanishing viscosity limit of Equation (2.1) holds.*

Proof. Because $\overline{u}^0$ is in $V \cap H^3(\Omega)$, u is in $L^\infty_{loc}([0, \infty); V \cap H^3(\Omega))$ (see, for instance, Theorem III.3.6, Remark III.3.7, and Theorem III.3.10 of [16]).

As observed in the proof of Theorem 6.1, $u \cdot \nabla u = \nabla q$ for some scalar field q. Then $\partial_t u + \nabla(p + q) = \nu \Delta u$ and taking the divergence of both sides

we conclude that $\Delta(p+q)=0$. Because of the radial symmetry, however, $p+q$ is constant on Γ and hence is constant on Ω. Thus, $\nabla(p+q)=0$ and $\partial_t u = \nu \Delta u$. But $u = 0$ on Γ so $\partial_t u = 0$ on Γ, and it follows that $\Delta u = \nu^{-1} \partial_t u$ is in V.

Because u is in V at time zero, we can use the expansion for ω in Equation (3.5) for all time, including for time zero. Because ω remains radially symmetric over time, the expansion reduces to

$$\omega(t, r, \theta) = \sum_{k=1}^{\infty} g_{0k}(t) \omega_{0k}(r) = \sum_{k=1}^{\infty} g_{0k}(t) C_{0k} J_0(j_{1k} r). \tag{6.2}$$

Since Δu is in V, it follows from Corollary 3.3 that $\Delta\omega$ has an expansion like that of Equation (6.2):

$$\Delta\omega(t, r, \theta) = \sum_{k=1}^{\infty} h_{0k}(t) C_{0k} J_0(j_{1k} r). \tag{6.3}$$

(The analogous expansion of $\Delta\omega$ including all the eigenvectors $\{\omega_{nk}\}$ fails to converge at $t=0$ for non-radially symmetric solutions because Δu is not, in general, in V.)

Since $\Delta J_0(j_{1k} r) = -\lambda_{0k} J_0(j_{1k} r)$, it follows that $h_{0k}(t) = -\lambda_{0k}(t) g_{0k}(t)$ and then by Equation (6.1) in strong vorticity form,

$$\partial_t \omega = \nu \Delta\omega, \tag{6.4}$$

that $g_{0k}'(t) = -\nu \lambda_{0k} g_{0k}(t)$. Thus, $g_{0k}(t) = g_{0k}(0) e^{-\nu \lambda_{0k} t}$ and

$$\omega(t, r, \theta) = \sum_{k=1}^{\infty} C_{0k} g_{0k}(0) e^{-\nu\lambda_{0k} t} J_0(j_{1k} r). \tag{6.5}$$

But then

$$\begin{aligned} \|\omega(t)\|_{L^2(\Omega)}^2 &= \sum_{k=1}^{\infty} C_{0k}^2 g_{0k}(0)^2 e^{-2\nu\lambda_{0k} t} \|J_0(j_{1k} r)\|_{L^2(\Omega)}^2 \\ &\le \sum_{k=1}^{\infty} g_{0k}(0)^2 \|C_{0k} J_0(j_{1k} r)\|_{L^2(\Omega)}^2 = \|\omega^0\|_{L^2(\Omega)}^2 \end{aligned} \tag{6.6}$$

and convergence in the vanishing viscosity limit follows from the condition in Equation (2.2) of Theorem 2.3. (Examining the argument in [6] shows that the convergence rate in Equation (2.1) is bounded by $C(\nu t)^{1/2}$.) □

We could have concluded the proof another, more indirect way, as follows. From Equation (6.5) we see that $\omega(t)$ is continuous in the $L^2(\Omega)$-norm at time zero. Thus, we can multiply Equation (6.4) by ω and integrate over time to give

$$\frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla\omega(t)\|_{L^2(\Omega)}^2 = \nu \int_{\Gamma} (\nabla\omega \cdot \mathbf{n}) \omega. \tag{6.7}$$

But $\nabla \omega = -(\Delta u)^\perp = 0$ on Γ because Δu is in V as we showed, and we conclude by integrating over time that ω is in $L^\infty([0, \infty); L^2(\Omega))$, and Equation (2.1) follows from the condition in Equation (2.2) of Theorem 2.3.

This argument shows that whenever the vorticity is continuous at time zero in the $L^2(\Omega)$-norm and $\Delta u(t)$ lies in V for all $t > 0$, Equation (2.1) holds. The latter condition, however, is very special. In fact, for $\overline{u}^0$ with regularity as in Theorem 6.2, $\partial_t u$ will always be in V, so $\Delta u = (\partial_t u + u \cdot \nabla u + \nabla p)/\nu$ is in V if and only if $u \cdot \nabla u + \nabla p$ is in V. This, in turn, can hold only if $\nabla p = -u \cdot \nabla u$ on Γ.

This suggests that for initial velocities for which $\omega(t)$ is continuous in the $L^2(\Omega)$-norm at time zero we might attempt to make an argument using Equation (6.7), though now without the right-hand side vanishing. This would require either control on $\nabla \omega$, as we had above, or on ω itself. (For solutions to (NS) with the boundary condition $\omega = u \cdot \mathbf{n} = 0$ rather than $u = 0$ this term would vanish and one can obtain the vanishing viscosity limit easily—though not using Theorem 2.3, which one would need to show applies to such solutions—as done, for instance, in [11] and [12].)

Another line of attack is also suggested by the proof of Theorem 6.2—to gain control on the coefficients $g(t)$ either by assumptions on the initial velocity or on the solution. Unless bounds as remarkably strong as those obtained in the proof of Theorem 6.2 are achieved, though, something more sophisticated must be employed to obtain Equation (2.1).

7. Interpretation in terms of length scales

In [3], Cheng and Wang consider the vanishing viscosity limit in the setting of a two-dimensional rectangular channel R, periodic in the x direction with period L and with characteristic boundary conditions (which include no-slip boundary conditions as a special case). They decompose any vector u on R of sufficient regularity as $u = \sum_{j=0}^\infty e^{2\pi ijx/L} u^j$ and define the projection $P_k u = \sum_{j=0}^k e^{2\pi ijx/L} u^j$ onto the space spanned by the first k modes. This in effect allows one to isolate successively finer-scale spatial variations in the direction tangential to the boundary. They then construct an approximation sequence $\{v^L\}$ to u by letting v^L be the solution to the equation that results after projecting each term in (NS) using P_N. (We have changed their notation somewhat.) Their v^L is the approximate-solution analog of the exact solution truncation represented by $\widetilde{u}^L$ in Equation (2.9).

The main result in [3] is that $v^{L(\nu)}$ converges to $\overline{u}$ in $L^\infty([0, T]; L^2(\Omega))$ as $\nu \to 0$. The requirement on $L(\nu)$ is the same as our condition on L in Equation (2.10) (with the additional condition that $L(\nu) \to \infty$ as $\nu \to 0$ as one would expect), so convergence of $v^{L(\nu)}$ to $\overline{u}$ occurs when only tangential length scales of order larger than ν are included in the approximations. (All length scales in the normal direction, however, are included. See Remark (5.1) concerning this issue in regards to the vorticity.)

The result in [3] makes an important observation about the difficulty of determining numerically whether or not the vanishing viscosity limit holds. Our method of decomposing the solution using the eigenfunctions of the Stokes operator, on the other hand, says little about computation, since approximating this decomposition numerically is probably as least as hard as approximating the solution itself. Nonetheless, it more directly characterizes the properties of the solution itself at different length scales.

The analog to the result in [3] is Theorem 5.5, which shows that Temam and Wang's conditions in Equation (2.5) and Equation (2.6), when applied only to the modes with tangential wavelengths of $CL(\nu)$ or higher, holds as long as the condition on N in Equation (2.10) hold. This does not, however, imply that $u^{L(\nu)}$ converges to $\overline{u}$ in the vanishing viscosity limit, only that if the vanishing viscosity limit fails to hold, the failure originates in the behavior of the tangential component of the gradient projected into the space spanned by the modes with tangential frequencies of order $L(\nu)$ or higher; that is, at length scales of order ν or lower.

The other conditions in Theorem 2.4 give alternative ways to measure the behavior of the solution at different length scales or frequencies. They show that we cannot simply say that if the vanishing viscosity fails to hold then the failure lies in the behavior of the solution at any particular range of length scales, but rather that the pertinent range of length scales varies with the measure of behavior. Whether any of these conditions brings us any closer to proving that the vanishing viscosity limit holds in general for smooth initial data in a bounded domain or to proving that it fails to hold in at least one instance remains completely unclear.

Appendix A. Bounds on the Eigenfunctions

In Lemma A.1 we state the basic identities involving the Bessel functions that we use. We then give a series of lemmas that lead to the bounds on the velocity and vorticity of the eigenfunctions in the boundary layer that we used in the proof of Theorem 2.4.

It is perhaps important to note that in the proofs that follow we avoid the use of asymptotic formulas for the Bessel functions, even when such formulas might appear to be useful. This is because we need to deal with the relative values of Bessel functions of different orders near a zero of one of the Bessel functions, and it is precisely in these situations that the errors in the asymptotic formulas dominate. Also, most of the following lemmas apply without change to their proofs with n being any nonnegative real value.

Lemma A.1. *For all nonnegative real numbers n and x,*

$$2nJ_n(x) - xJ_{n-1}(x) = xJ_{n+1}(x), \tag{A.1}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \tag{A.2}$$

$$J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x), \tag{A.3}$$

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x), \tag{A.4}$$

$$\frac{x^n J_n(\alpha x)}{\alpha} = \int x^n J_{n-1}(\alpha x)\, dx, \tag{A.5}$$

$$J_n(\alpha x) x^{-n} = -\alpha \int J_{n+1}(\alpha x) x^{-n}\, dx, \tag{A.6}$$

$$(\beta^2 - \alpha^2) \int x J_n(\alpha x) J_n(\beta x)\, dx = x \left[\alpha J_n'(\alpha x) J_n(\beta x) - \beta J_n'(\beta x) J_n(\alpha x)\right], \tag{A.7}$$

$$\int x J_n(ax)^2\, dx = \frac{1}{2}\left[x^2 J_n'(ax)^2 + \left(x^2 - \frac{n^2}{a^2}\right) J_n(ax)^2\right], \tag{A.8}$$

$$\int x J_n(ax)^2\, dx = \frac{x^2}{2}\left[J_n(ax)^2 - J_{n-1}(ax) J_{n+1}(ax)\right]. \tag{A.9}$$

Proof. These are standard identities for Bessel functions. For instance, see Equations (6.28), (6.29), (6.30), (6.31), (6.38), (6.39), (6.51), (6.52), and (6.53) of [2]. □

Lemma A.2. *For all nonnegative integers n and all positive integers k,*

$$1 < j_{n+1,k} - j_{nk} < \frac{\pi}{2},$$

where j_{nk} is defined in Equation (4.4).

Proof. Let $j_{\nu k}$ be the k-th positive zero of J_ν, where we now allow ν to be a real number in the interval $[0, \infty)$. It is shown in [5] and [4] that for all $k \geq 1$, $j_{\nu k}$ is strictly concave as a function of ν and that $dj_{\nu k}/d\nu > 1$ (see also [9]). Thus, the function $n \mapsto j_{n+1,k} - j_{nk}$ is strictly decreasing as a function of n. But by Equation (2.9) of [4], $j_{n+1,k} - j_{nk} \to 1$ as $n \to \infty$, so $j_{n+1,k} - j_{nk} > 1$.

The positive zeros of J_0 lie in the intervals $(m\pi + \frac{3}{4}\pi, m\pi + \frac{7}{8}\pi)$, $m = 0, 1, \dots$, and the positive zeros of J_1 lie in the intervals $(m'\pi + \frac{1}{8}\pi, m'\pi + \frac{1}{4}\pi)$, $m' = 1, 2, \dots$. That the zeros lie in only these intervals is shown in Section 15.32 p. 489 and Section 15.34 p. 491 of [18] using an approach of Schafheitlin's. That each of these intervals contains at least one zero is shown on p. 104 of [2]. But $j_{1k} - j_{0k} > 1$ as we showed above so each interval contains precisely one zero. Because the zeros of J_0 and J_1 are interleaved (see p. 106 of [2], for instance) we can then conclude that $j_{1k} - j_{0k} < \frac{\pi}{2}$. But as we observed above, the function $n \mapsto j_{n+1,k} - j_{nk}$ is strictly decreasing as a function of n, so $j_{n+1,k} - j_{nk} < \frac{\pi}{2}$ holds for all $n \geq 0$. □

Lemma A.3. *For all $n = 0, 1, \dots$ and $k = 1, 2, \dots,$*

$$n + k < j_{nk} < \pi(n/2 + k) \le \pi(n + k).$$

Proof. By Lemma A.2, for all n and j,

$$j_{nk} = j_{0k} + \sum_{m=1}^{n} (j_{mk} - j_{m-1,k}) \ge j_{0k} + n > n + k,$$

because $j_{0k} > k$ (which follows directly from Equation (4.3); see p. 485-486 of [18], for instance). By an observation in the proof of Lemma A.2 it follows that $j_{0k} < \pi k$, and a similar argument using the inequality $j_{n+1,k} - j_{nk} < \frac{\pi}{2}$ from Lemma A.2 gives the upper bound on j_{nk}. □

Lemma A.4. *Let $\alpha = j_{n+1,k}$ and $\beta = j_{nk}$. For $n = 0, 1, 2, \dots$ and $k = 1, 2, \dots,$*

$$\left| \frac{J_n(\alpha x)}{J_n(\alpha)} \right| \le 1 \text{ if } \frac{\beta}{\alpha} < x < 1.$$

Proof. Let $g(x) = J_n(\alpha x)/|J_n(\alpha)|$. From Equation (A.4), $J_n'(\alpha) = (n/\alpha) J_n(\alpha)$, so $J_n'(\alpha)$ has the same sign as $J_n(\alpha)$. From this we conclude that $|g|$ is increasing in a left-neighborhood N of 1.

Between each zero of J_n there is exactly one zero of J_{n+1} (see p. 106 of [2], for instance). Between each zero of J_n there is also exactly one zero of J_n', because the maximum values of J_n are all positive and the minimum values are all negative (see, for instance, p. 107 of [2]) and J_n' has no repeated positive roots (this follows from the defining equation Equation (4.3)). Thus, the neighborhood N includes all x such that $\beta < \alpha x < \alpha$. Since $|g(1)| = 1$ it follows that $|g(x)| \le 1$ for all such x. □

Lemma A.5. *Let $\alpha = j_{n+1,k}$ and $\beta = j_{nk}$. There exists a constant C such that for all $n = 0, 1, \dots$ and $k = 1, 2 \dots n$,*

$$\left| \frac{J_{n+1}(\alpha x)}{J_n(\alpha)} \right| \le C n (1 - x) \text{ if } \frac{\beta}{\alpha} < x < 1.$$

Proof. Since $J_{n+1}(\alpha) = 0$, Equation (A.5) with $n + 1$ in place of n gives

$$J_{n+1}(\alpha x) = -\frac{\alpha}{x^{n+1}} \int_x^1 t^{n+1} J_n(\alpha t) \, dt.$$

As long as $\beta < \alpha x < \alpha$, $J_n(\alpha t)$ does not change sign on the interval $(x, 1]$ and has its maximum value on this interval at 1, as observed in the proof of Lemma A.4. Thus,

$$|J_{n+1}(\alpha x)| \le \frac{\alpha}{x^{n+1}} |J_n(\alpha)| \int_x^1 t^{n+1} \, dt \le \frac{\alpha \, |J_n(\alpha)|}{(\beta/\alpha)^{n+1}} (1 - x).$$

But by Lemma A.2 and Lemma A.3,

$$1 - \frac{\beta}{\alpha} = \frac{\alpha - \beta}{\alpha} \le \frac{\pi/2}{n+2} \implies \frac{\beta}{\alpha} \ge 1 - \frac{\pi}{2n+4}$$

so

$$(\beta/\alpha)^{-(n+1)} \le \left(1 - \frac{\pi}{2n+4}\right)^{-(n+1)} \le e^{\pi/2},$$

the last inequality following from elementary calculus. We conclude that

$$|J_{n+1}(\alpha x)| \le Cn\,|J_n(\alpha)|\,(1-x), \tag{A.10}$$

which completes the proof. □

Lemma A.6. *Let $\alpha = j_{n+1,k}$ and $\beta = j_{nk}$. There exists a constant C such that for all $n = 0, 1, \dots$ and $k = 1, 2 \dots, n$,*

$$\left|\frac{J_{n-1}(\alpha x)}{J_n(\alpha)}\right| \le C \text{ if } \frac{\beta}{\alpha} < x < 1.$$

Proof. Because the positive zeros of J_{n-1} are interlaced with those of J_n, J_{n-1} does not change sign on the interval $[\beta, \alpha]$. From Equation (A.4) with $n-1$ in place of n, $J'_{n-1}(\beta) = ((n-1)/\beta)J_{n-1}(\beta)$, so $J'_{n-1}(\beta)$ has the same sign as $J_{n-1}(\beta)$, and we conclude that J_{n-1} reaches it maximum value on the interval $[\beta, \alpha]$ at β. Therefore, for $\beta/\alpha < x < 1$,

$$\left|\frac{J_{n-1}(\alpha x)}{J_n(\alpha)}\right| \le \left|\frac{J_{n-1}(\beta)}{J_n(\alpha)}\right|.$$

But, by Equation (A.1), $J_{n+1}(\beta) = 2(n/\beta)J_n(\beta) - J_{n-1}(\beta) = -J_{n-1}(\beta)$, so

$$\left|\frac{J_{n-1}(\alpha x)}{J_n(\alpha)}\right| \le \left|\frac{J_{n+1}(\beta)}{J_n(\alpha)}\right| \le Cn(1 - \beta/\alpha) \le C,$$

where we used Lemma A.5 and Lemma A.3. □

Lemma A.7. *We have $\|\omega_j\|^2_{L^2(\Gamma_\delta)} \le 2\delta$ when $\delta \le \lambda_j^{-1/2}$.*

Proof. Let $\omega_j = \omega_{nk}$ and $\alpha = j_{n+1,k} = \lambda_j^{1/2}$. Then

$$\|\omega_j\|^2_{L^2(\Gamma_\delta)} = 2\pi C_{nk}^2 \int_{1-\delta}^1 r J_n(\alpha r)^2\, dr = 2\int_{1-\delta}^1 r\frac{J_n(\alpha r)^2}{J_n(\alpha)^2}\, dr.$$

In the integrals above, with $\beta = j_{nk}$,

$$\beta/\alpha = 1 - (\beta - \alpha)/\alpha \le 1 - 1/\alpha = 1 - \lambda_j^{-1/2} \le 1 - \delta < r < 1,$$

where we used Lemma A.2, and the lemma follows from Lemma A.4. □

Employing Lemma A.7, we can extend its range of applicability, though with a higher bound on the width of the boundary layer.

Lemma A.8. *For all $n = 0, 1, \dots$, $k = 1, \dots, n$, and all $\delta < 2\pi\lambda_{n1}^{-1/2}$,*

$$\|\omega_{nk}\|^2_{L^2(\Gamma_\delta)} \le 2\delta.$$

Proof. It follows from Lemma A.3 that $j_{n,n}/j_{n,1} \le \pi(n+n)/(n+1) \le 2\pi$; the lemma follows from this inequality and Lemma A.7. □

Remark A.1. It is possible to extend Lemma A.8 to include all values of k. The idea of the proof is that for $k > n$, ω_{nk} passes through k complete half-periods (annuli in the unit disk lying between successive nonnegative zeroes of $J_n(j_{n+1,k}r)$) and ends with a partial period. Since $J_n(x)$ decays like $x^{1/2}$ and the spacing between consecutive zeros of J_n approaches a constant, the L^2-norms of ω_{nk} on each of those half-periods converges to a constant, and since the L^2-norm of ω_{nk} on the entire unit disk is 1, the square of the L^2-norm of ω_{nk} on the last half-period is less than C/k (with C near 1). But the last half-period has a width that is greater than C/k. Extending this argument to m periods, what we have shown is that

$$\|\omega_{nk}\|^2_{L^2(\Gamma_{Cm/k})} \le m/k.$$

With the assumed bound on δ, we choose m so that m/k is of the same order as δ, and the proof is essentially complete.

Lemma A.9. *There exist positive constants C_1 and C_2 with $C_2 < 1$ such that for all $n = 0, 1, \dots$ and all $k = 1, \dots, n$,*

$$\|u_{nk}\|^2_{L^2(\Gamma_\delta)} \le C_1\delta^3$$

when $\delta < C_2\lambda_{n1}^{-1/2}$.

Proof. In the proof that follows, we will often use Lemma A.3 without explicit mention.

Let $\alpha = j_{n+1,k}$. We bound first the radial component of u_{nk}. We have,

$$\frac{J_n(\alpha r) - J_n(\alpha)r^n}{J_n(\alpha)r} = r^{n-1}g_n(r),$$

where

$$g_n(r) = \frac{J_n(\alpha r)r^{-n}}{J_n(\alpha)} - 1 = -\frac{\alpha}{J_n(\alpha)}\int_r^1 \frac{J_{n+1}(\alpha x)}{x^n}\,dx \quad = -\alpha\int_r^1 \frac{B_{nk}(x)}{x^n}\,dx,$$

and

$$B_{nk}(x) = \frac{J_{n+1}(\alpha x)}{J_n(\alpha)}.$$

To verify the second equality we use the identity in Equation (A.6), from which it follows that the second expression for g_n is

$$\frac{J_n(\alpha r)r^{-n} + C}{J_n(\alpha)}$$

for some constant C. But all three expressions for g_n are zero at $r = 1$, so we have the correct limits of integration in the second expression. It follows from Lemma A.5 and our third expression for g_n that

$$|g_n(r)| \le Cn\alpha\int_r^1 x^{-n}(1 - x)\,dx \le \frac{Cn^2}{r^n}(1 - r)^2$$

for all $1 - r \le 1/\alpha$.

From Equation (4.8),

$$u_{nk}^r(r,\theta) = \frac{g(r)r^{n-1}}{\alpha^2\pi^{1/2}} ine^{in\theta}\widehat{e}_r,$$

so when $\delta \le 1/\alpha$ we have

$$\begin{aligned}\|u_{nk}^r\|_{L^2(\Gamma_\delta)}^2 &= 2\pi \int_{1-\delta}^1 r\left|u_{nk}^r\right|^2 dr \le \frac{Cn^6}{\alpha^4}\int_{1-\delta}^1 \frac{(1-r)^4}{r^{2n-1}}\,dr \\ &\le Cn^2(1-\delta)^{1-2n}\delta^5 \le C\delta^3.\end{aligned}$$

In the last inequality we used

$$\delta < C_2\lambda_{n1}^{-1/2} = \frac{C_2}{j_{n+1,k}} \le \frac{C_2}{n+1}$$

so

$$\begin{aligned}(1-\delta)^{2n-1} &\ge \left(1 - \frac{C_2}{n+1}\right)^{2n-1} = (G(C_2, n+1))^2\left(1-\frac{C_2}{n+1}\right)^{-3} \\ &\ge (1-C_2)^2(1-C_2)^{-3} = (1-C_2)^{-1} = C > 0,\end{aligned}$$

where G is the function of Lemma A.11.

For the angular component of u_{nk}, we write

$$\begin{aligned}&\alpha J_{n+1}(\alpha r) - \alpha J_{n-1}(\alpha r) + 2nJ_n(\alpha)r^{n-1} \\ &\quad= [2nJ_n(\alpha r) - \alpha r J_{n-1}(\alpha r)] + \alpha r J_{n-1}(\alpha r) - 2nJ_n(\alpha r) \\ &\qquad\qquad + \alpha J_{n+1}(\alpha r) - \alpha J_{n-1}(\alpha r) + 2nJ_n(\alpha)r^{n-1} \\ &\quad= \alpha(r+1)J_{n+1}(\alpha r) + 2n\left[J_n(\alpha)r^{n-1} - J_n(\alpha r)\right] \\ &\qquad\qquad + \alpha J_{n-1}(\alpha r)(r-1).\end{aligned}$$

From Equation (4.8) we then have

$$\begin{aligned}|u_{nk}^\theta|^2 &\le C\frac{\alpha^2(r+1)^2}{\alpha^4}\frac{J_{n+1}(\alpha r)^2}{J_n(\alpha)^2} + C\frac{n^2}{\alpha^4}\left(\frac{J_n(\alpha)r^{n-1} - J_n(\alpha r)}{J_n(\alpha)}\right)^2 \\ &\qquad + C\frac{\alpha^2 J_{n-1}(\alpha r)^2(1-r)^2}{\alpha^4 J_n(\alpha)^2} \\ &\le C(1-r)^2 + \frac{C}{n^2}\left(r^{n-2}g_{n-1}(r)\right)^2,\end{aligned} \tag{A.11}$$

where we applied both Lemma A.5 and Lemma A.6.

The first term in Equation (A.11) contributes no more

$$C\int_{1-\delta}^1 r(1-r)^2\,dr \le C\int_{1-\delta}^1 (1-r)^2\,dr \le C\delta^3,$$

and the same is true of the second term in Equation (A.11) arguing as for u_{nk}^r, and this completes the proof. □

Lemma A.10. *When $m \ne n$, $\langle u_{mj}, u_{nk}\rangle_{L^2(\Gamma_{c\nu})} = \langle \omega_{mj}, \omega_{nk}\rangle_{L^2(\Gamma_{c\nu})} = 0$.*

Proof. We have,

$$\langle \omega_{mj}, \omega_{nk} \rangle_{L^2(\Gamma_{c\nu})} = \int_{1-c\nu}^{1} r f(r) \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta \, dr,$$

where $f(r)$ is a product of two Bessel functions. When $m \neq n$, the inner integral is zero. A similar argument gives $\langle u_{mj}, u_{nk} \rangle_{L^2(\Gamma_{c\nu})} = 0$. □

Lemma A.11. *Let α be in $(0, 1)$ and define $G_\alpha : [1, \infty) \to [0, \infty)$ by*

$$G_\alpha(x) = \left(1 - \frac{\alpha}{x}\right)^x.$$

Then for all $x > 1$,

$$1 - \alpha \leq G_\alpha(x) < e^{-\alpha}.$$

Proof. The proof is elementary. □

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